ARTICLES

Central Problems and Directions in the Philosophy of Mathematics

It would certainly not be excessive to claim that practically all human beings have, at one time or another, some kind of involvement with mathematics. For most, this sharing in the mathematical experience takes such simple forms as counting or measuring when they are involved in such ordinary and everyday activities as, for example, buying and selling. Some, in addition, have the opportunity of some formal contact with mathematics in an academic context, at school, college or university. A very few, proportionately speaking, become professional mathematicians, as teachers of mathematics or researchers in pure mathematics or as employers of applied mathematics in a wider professional context such as, for example, architecture or engineering. But very few indeed, finally, concern themselves with the task of reflecting upon the nature of mathematics, with trying to understand and furnish an explanation of mathematical activity. These last are involved not merely with mathematics but precisely with a philosophical consideration of mathematics, with the *philosophy of mathematics*.

Even though, for obvious reasons, a philosophical consideration of mathematics has to make constant reference to, and indeed can never allow itself to lose sight of, both particular instances of mathematical reasoning and particular cases taken from the history of mathematics, it does not deal directly, let alone for their own sake, with either of these. The philosophy of mathematics is accordingly not to be confused, let alone identified, with either mathematics itself or with the study of the history of mathematics. Rather, the direct concern of the philosophy of mathematics is the multiplicity of precisely *philosophical* perspectives and problems, presuppositions and consequences, which have been and are either taken for granted or implied by mathematical reasoning and which have arisen or played a role within the continuing history of the development of mathematical thought.

Furthermore, the general philosopher, as opposed to the specialist in the philosophy of mathematics, will not be interested even in these narrowly philosophico-mathematical issues for their own sake and in their own right. Rather, the general philosopher will be concerned with the issues directly considered by the philosophy of mathematics in so far as these issues and the various combinations of philosophical responses, the diverse "philosophies of mathematics", which have been proposed to resolve them have, since the time of the pre-Socratics and especially the Pythagoreans, always played an important role and even exercised a determinative influence with respect to the development of the conception of both the nature and the peculiar problematic of philosophy itself. Indeed, the Western tradition in philosophy, that tradition which has its historical point of departure in the thought of the pre-Socratics, has always advocated, or at the very least either presupposed or implied, one or more, more or less explicit, philosophical interpretations of the objects, the nature and the epistemological value of mathematical knowledge. This constantly present philosophical understanding and evaluation of mathematical knowledge has itself always had extremely important repercussions, and has thereby exercised a fundamental influence, upon the development experienced by this tradition and the ideals and goals which it constantly proposed to itself.

Accordingly, mathematical knowledge has generally been considered by the Western philosophical tradition to be the very paradigm, the one indisputable instance, of authentic knowledge. Mathematical knowledge has continually, at least up to the present century, been held to represent the one unquestionable realization of the *Platonic* ideal of *episteme*, of genuine *scientific* knowledge. That is, of knowledge which is characterized by certainty insofar as it is supposed to be knowledge of the eternal, the immutable and necessary. Knowledge, which represents the grasp of laws which are determinative with respect to reality, knowledge which is formulated in terms of propositions which are truly universal and necessary — propositions whose truth may be discerned in an entirely *a priori* manner and wholly independently of sense perception. This conception, this philosophy of mathematics, which has largely dominated the Western philosophical tradition until very recently and which has by no means as yet entirely lost its influence, has generally been seen as providing a model of what philosophical knowledge itself (as well as scientific and even theological knowledge) ought to be, or at least ought to strive to be: a purely axiomatic-deductive system grounded in the *a priori* seizure of evident and apodictic first principles.

Furthermore, the general philosopher will be concerned with the issues which are dealt with directly by the philosophy of mathematics not only insofar as these various issues exercise an influence on the conception of philosophy itself but also insofar as they present him with a series of test-cases in terms of which he will be able to evaluate the various philosophical theories which he advocates or adheres to in other parts of theoretical philosophy. Thus, the general philosopher will be able to test the coherence of the positions which he proposes, for example, in metaphysics, the theory of knowledge, the philosophy of mind and especially in philosophical logic, by determining to what extent they can do justice and contribute to the resolution of explicitly philosophico-mathematical issues.

I shall now attempt to introduce the reader to the philosophy of mathematics from two points of view: (1) in the first place I shall present a preliminary introduction to the problematic considered by the philosophy of mathematics from a theoretical or systematic point of view; (2) in the second place I shall present a synthetic introduction to the problematic considered by the philosophy of mathematics from the historical point of view. Of course these two aspects, neither of which can be neglected by any philosophical consideration of mathematics, are intimately connected and I only separate them in the present context for the sake of expository convenience.

(1) THEORETICAL INTRODUCTION TO THE PHILOSOPHY OF MATHEMATICS

One might well expect that a suitable point of departure for a theoretical or systematic introduction to the problematic considered by the philosophy of mathematics is to be attained by means of the provision of at least a preliminary or nominal definition of mathematics which constitutes the problematic considered by the philosophy of mathematics; the problem of determining just what mathematics is and, to begin with, of determining just how we are going to be able to do so.

It is possible, nonetheless, to give a few preliminary indications.

1. It is possible to indicate the etymology of the word "mathematics". It is derived from the Greek word *mathano*, to learn, and the connected *mathema* which means a teaching or doctrine or discipline. The term "mathematics" originally meant simply a body of knowledge which could be taught and learnt.

2. It is also possible to enumerate the various disciplines, branches of mathematics and theories, which are generally acknowledged as being parts of mathematics. To mention but a few (from well over two hundred!): (1) *Arithmetic*: Theory of numbers, combinatory analysis, theory of groups and theory of sets. (2) *Algebra*: Probability calculus, infinitesimal calculus (integral and differential), functions of real variables, functions of complex variables, differential equations, calculus of variations, functional calculus, integral equations. (3) *Geometry*: Elementary (Euclidean) geometry, axiomatic (non-Euclidean) geometries, projective and descriptive geometries, topology, algebraic (analytic) and differential geometries. (4) *Mathematical* or *Symbolic Logic*: That is, formal logic treated algebraically and by means of a symbolic notation. (But is mathematical logic a part of mathematics or is, rather, mathematics a part of logic? Here we are entering

into an extremely controversial issue which has been central to the philosophico-mathematical problematic of the last one hundred years!)

3. It is possible to indicate various, indeed innumerable, experiential contacts with practical mathematics which are part of day-to-day life, contacts with theoretical mathematics in an academic setting as part of non-specialist or general education, professional and specialist contacts with mathematics, pure and applied, which are proper to working mathematicians and teachers of mathematics. It is important to be aware of the fact that most contemporary professional mathematicians are at the present time prepared to go no further than to "define" mathematics as being quite simply (in a misleadingly unproblematic manner) as *"that activity which is carried out by mathematicians."*

Let us now return to our initial remark to the effect that it is not possible to give even a preliminary or merely nominal definition of mathematics. Just why is this so?

1. Because it is not possible simply to take over some inherited, historically given, definition of mathematics. For example, the "classical" definition of mathematics as the "science of quantity" which is divided into the two subordinate disciplines of arithmetic and geometry which deal with the two species of quantity, multitude and magnitude, respectively. Another example is the definition of arithmetic as the "science of quantity" and of geometry as "the science of space and its properties." Another example is the definition of mathematics as "the science of the infinite." A final example is that of the definition of mathematics as the "science of formal or purely axiomatic systems". Now, the point is that not one of these proposed definitions of mathematics can simply be taken for granted and adopted because each and every one of them has been repeatedly called into question and, more often than not, has been found wanting. There is at yet no, non-trivial, definition of mathematics which is acceptable to all methematicians, let alone to all philosophers of mathematics.

2. Because even our etymological observation to the effect that mathematics is a discipline, a body of knowledge that can be taught and learnt does not enable us to arrive at a definition of mathe-

matics. In fact it is precisely in our recognition of mathematics as a *body of knowledge* that can be taught and learnt and passed on from generation to generation as a socio-cultural deposit, that the philosophico-mathematical problematic emerges most clearly and as extremely arduous.

It is possible to express the philosophical problematic which arises from a consideration of mathematics as a body of knowledge in the form of, at least at first sight but deceptively, three simple questions:

1. Just *what* is it that we know in mathematics? That is, just what is mathematical knowledge *about*? What is the nature of the *objects* which constitute its subject matter? Of what kind of "thing", if of anything at all, is mathematics the knowledge?

2. Just *how* do we know whatever it might be that we have knowledge of in mathematics? What is the *manner* of mathematical knowledge? By what means do we acquire mathematical knowledge?

3. Just what *kind* of knowledge is the knowledge that we have of whatever it might be that we know in mathematics? What is the *nature* of mathematical knowledge? What is the value, the cognitive peculiarities and characteristics of mathematical knowledge?

Let us now take each of these misleadingly simple questions in turn and very briefly indicate just how complex and difficult they really are. Indeed, not one of these questions can be resolved at the present time in a manner which could be satisfactory to all concerned. This is not only because they are such difficult and complex questions but also because they are *philosophical* rather than directly mathematical questions. As such, their resolution presupposes not merely a knowledge of mathematics and an acquaintance with its history but also the prior adoption of some wider philosophical frame of reference or theoretical stance which, in its turn, demands justification. Accordingly, to become involved with these questions represents a movement beyond the narrow confines of mathematics itself and an entry into philosophy. In fact the greater part of professional mathematicians rarely bother themselves with these questions which, insofar as their resolution is of no practical importance for the mastery of mathematical techniques and their wider applications, they either ignore or willingly leave to the philosopher of mathematics.

One should note that these three questions are intimately connected with each other. The kind of answer which is given to any one of them will be wholly determinative with respect to, and in its turn will be determined by, the kind of answer which is given to the other two.

1. The first question: Just *what* is it that we know in mathematics, what is mathematical knowledge *about?* This question concerns itself with the identification and characterization of the *objects* which constitute the subject-matter of mathematics. At first sight the answer to this question might appear to be somewhat obvious. Mathematics, as everyone "knows", is about, has as its objects or subject-matter, numbers, sets, classes, groups, functions, properties, relations, figures and so on. But just what is the nature of these, variously called, mathematical "objects", "entities", "concepts", "categories", "structures"? This question thus raises the complex problem of the determination of the *ontological status* of the subject-matter of mathematics. Several alternative, and indeed radically opposed, possibilities immediately present themselves:

1.1 The objects which constitute the subject-matter of mathematics are *entities,* some kind of realities, in their own right. They accordingly are endowed with a peculiar, "ideal", ontological status whereby they are completely transcendent and independent with respect to the actually existing, contingent world — including the human mind or consciousness which, in its mathematical activities, is somehow able to grasp or intuit them but which neither invents them nor constructs them but solely *discovers* them. If one were to adopt this alternative then one would be advocating what is human as a "realist" and at times also as a "Platonist" philosophical position with respect to the problem of the ontological status of mathematical objects.

1.2. The objects which constitute the subject-matter of mathematics are not entities in their own right but simply abstract concepts. Their ontological status is that of being no more than mental contents which, as such, exist only in thought, that is, in the mind or consciousness which forms them by means of a process of formal abstraction which has its point of departure in the operation of sensibility. Although they are no more than mental concepts the objects of mathematics are no mere arbitrary fictions — they have an objective foundation in reality: the quantitative accidental forms of figure and number which inhere in existing material things and which are indirect objects of the external senses or "common sensibles". If one were to adopt this alternative (which corresponds to the Thomistic understanding of the nature of the objects of mathematics) then one would be advocating what is known as a "moderate realist" philosophical position with respect to the problem of the ontological status of mathematical objects.

1.3 The objects which constitute the subject-matter of mathematics are, for this alternative as well, simply abstract concepts. Their ontological status is that of being no more than mental contents which, as such, exist only in thought, that is, in the mind which forms them. But, and in contrast with the previous alternative, they are formed by means of a process of construction or definition which is entirely *a priori* in character and radically different from any kind of process of abstraction which has its point of departure in the operation of sensibility. If one were to adopt this position then one would be advocating what is known as an "intuitionist" or "constructivist" philosophical position with respect to the problem of the ontological status of mathematical objects — a position which is ultimately Kantian in inspiration.

1.4 The objects which constitute the subject-matter of mathematics are no more than mere symbols whose manipulation is determined by formal axiomatics and which, of themselves, do not denote anything at all — so that it would be quite correct to say that mathematical knowledge does not have any objects, is not really about anything. These symbols, and the rules which govern their employment, are the result of some process of definition which is entirely arbitrary or conventionalistic in character. If one were to adopt this alternative then one would be advocating what is known as a "formalist" philosophical position with respect to the ontological status of mathematical objects.

1.5 The objects which constitute the subject-matter of mathematics are, as with the preceding alternative, mere symbols whose manipulation is determined by formal axiomatics and which, of themselves, do not denote anything at all. In this case also then it would be quite correct to say that mathematical knowledge does not have any objects, that it is not really about anything. But, and in contrast with the previous alternative, these symbols, and the rules which govern their employment, are not at all the result of some process of definition which is either arbitrary or merely conventionalistic in character. Rather, they are defined through derivation from the basic concepts and laws of another discipline — formal logic. If one were to adopt this alternative then one would be advocating what is known as a "nominalistic logicist" position with respect to the ontological status of mathematical objects.

We see therefore that our first question "what is mathematical knowledge about, what is the nature of the objects of mathematics?" poses an extremely complicated problematic. It is a problematic which is not satisfactorily resolved by any of the proposed solutions each of which in fact, although we cannot enter into the details of this matter at this point, reveals itself as frought with difficulties. Contemporary writings on the philosophy of mathematics usually refer to this problematic of the ontological status of the objects of mathematical knowledge as that of "mathematical reality" and also as that of "mathematical existence".

2. The second question: Just *how* do we know whatever it might be that we have knowledge of in mathematics? By what means do we attain mathematical knowledge? Once again several alternative, and indeed radically opposed, possibilities present themselves. One should note the theoretical continuity which holds between these possibilities and the various corresponding responses which had been presented to the first question:

2.1 We know mathematical objects (in the sense of transcendent entities) by means of a purely intellectual or supersensible intuition which, whilst it might find useful illustrations in the data and especially in the imaginative pictorial representations which are furnished to the mind by the operation of sensibility is nonetheless radically indepedent of it.

2.2 We know mathematical objects (in the sense of abstract concepts which have a foundation in reality) by means of a process of formal abstraction which has its point of departure and is grounded in the operation of sensibility.

2.3 We know mathematical objects (in the sense of abstract concepts of which, in line with the Kantian doctrine of the unknowability of reality in itself, we can neither affirm nor deny that they have a foundation in reality) as the result of a finite process of construction or definition which, insofar as it has its point of departure in a peculiar and original intuition of the natural numbers, is wholly independent of the operation of sensibility and is to be characterized as being carried out exclusively *a priori*.

2.4 We know mathematical objects (in the sense of non-denoting symbols whose manipulation is governed by some formal axiomatic) as the result of a process of construction or definition which is entirely independent of the operation of sensibility and is thus carried out in an exclusively *a priori* manner which is further to be characterized as being entirely arbitrary or at the least determined by convention.

2.5 We know mathematical objects (in the sense of non-denoting symbols) as the result of a process of construction or definition which is entirely independent of the operation of sensibility and is thus carried out in an exclusively *a priori* manner but which is neither arbitrary nor determined by mere convention because it is ultimately grounded in and derived from the basic concepts and laws of formal logic.

We see therefore that also our second question "just how do we come to know mathematical objects (whatever they might be)?" also poses an extremely complicated problematic. A problematic which is clearly intimately connected with that posed by our first question and which, like it, is not satisfactorily resolved by any of the proposed solutions. Contemporary writings on the philosophy of mathematics usually refer to this problematic of the manner of our knowledge of mathematical objects as that of "mathematical intuition" and also as that of "mathematical access".

3. The third question: Just what *kind* of knowledge do we have of the objects of mathematics? Just what is the nature of

mathematical knowledge? This question raises the problem of the epistemological status of mathematical knowledge. Once again several alternative, and indeed radically opposed, possibilities present themselves. These, although intimately connected with the responses which had presented themselves with respect to the two previous questions, do not lend themselves as easily to a straightforward five-fold classification. Hence, at this point, it will be best to simply allude to some of these alternatives by means of the enumeration of a selection of representative questions which can be raised about the epistemological character of the *propositions* whereby we express our mathematical knowledge.

3.1 Are mathematical propositions *a priori* or *a posteriori?* That is, can their truth-value be determined independently of or only on the basis of some reference to sense experience? Are they to use Leibniz's terminology, *truths of reason* or *truths of fact?*

3.2 Are mathematical propositions *analytic* or *synthetic*? That is, in terms of Kant's understanding of these classifications, are they propositions which do or do not contain the concept of their predicate within the concept of their subject; are they merely explicative (mere tautologies) or authentically informative? Or, to employ that other understanding of these classifications which has received greater attention especially since Frege. are they propositions which are or are not true in virtue of their logical form alone?

3.3 If mathematical propositions are deemed to be analytic *a priori* does this necessarily mean that they are no more than mere tautologies or might they, despite their analytic status, still be authentically informative? But then, just what understanding of the analytic-synthetic distinction is to be presupposed by any attempt to answer this question?

3.4 If mathematical propositions are deemed to be analytic; then they will be recognized as being also *a priori*, but if they are deemed to be synthetic are they synthetic *a priori* or synthetic *a posteriori?*

3.5 If mathematical propositions are deemed to be synthetic *a priori* is this because they are supposed to be grounded in a purely intelligible, supersensible, intuition of mathematical objects (taken in the sense of transcendent entities) or because they are supposed to be grounded in the procedure of a purely *a priori* construction through definition of mathematical objects (taken as abstract concepts) ?

3.6 If, on the other hand, mathematical propositions are deemed to be synthetic *a posteriori,* mere empirical generalizations grounded in sense perception (be it the external perception of sensible objects or the inner perception of mental states) how is then one to account for the necessity, universality and certainty that the Western philosophical tradition has generally attributed to mathematical propositions? But then, should one simply take for granted that this tradition has in fact been correct in attributing these characteristics to mathematical propositions? Are they not perhaps contingent rather than necessary, particular or at the most mere generalizations rather than authentically universal, dubitable rather than certain?

3.7 Finally, when we affirm mathematical propositions to be truths, to be true, is this to be taken in some kind of absolute sense (to give but one example, because they would be supposedly descriptive of and thus enter into a relationship of correspondence with mathematical objects taken as transcendent entities) or not? Or perhaps is their characterization as *truths* not be taken in any kind of absolute sense (according to which they would be neither true nor false) but as indicating no more than, for example, that they are internally consistent (lacking in internal contradiction) as the result of some arbitrary definition of the terms which appear in them so that they would be "truths by convention"? But then, even in this case, should one simply take it for granted that *all* mathematical propositions must be either true or false, must have a determined truth-value, no matter what sense one attributes to the term "truth"?

We see therefore that also our third question "what is the nature of mathematical knowledge?" poses an extremely complicated problematic which is constituted by a set of almost inextricably inter-twined problems and perspectives of which the representative questions just presented do no more than give an initial indication. Contemporary writings on the philosophy of mathematics usually refer to this problematic of the epistemological status of mathematical knowledge as that of "mathematical truth".

In the present context we cannot enter into the various arguments which could be presented for and against the alternatives mentioned above with respect to the three questions as to the objects, the manner and the nature of mathematical knowledge. To consider these various arguments is precisely the task of the philosophy of mathematics. But I should like to make a few general remarks which are relevant to an appreciation, from a theoretical and systematic point of view, of the problematic which is the concern of the philosophy of mathematics:

1. The problematic which is the concern of the philosophy of mathematics and which was delineated above in terms of the three questions as to the objects, the manner and the nature of mathematical knowledge closely parallels, and might even be considered as a particular instance of, the *classical philosophical problems of universals.* Consequently it is possible to classify diverse philosophies of mathematics (which are constituted basically by different combinations of the possible alternative responses to the three questions indicated) by analogy with the various philosophical positions on the problem of universals as "realist" or "moderate realist" or "conceptualist" or "nominalist". To give but one example, if one responds to the first question as to the ontological status of mathematical objects by saying that they are transcendent entities and if one responds to the second question as to the manner of our knowledge of mathematical objects (taken as transcendent entities) by saying that the mind grasps them by means of a purely intellectual intuition, by means of which it neither invents nor constructs but solely discovers them, and if one responds to the third question as to the nature of mathematical knowledge by saying that mathematical propositions are absolute truths because they correspond to and are descriptive of mathematical objects (taken as transcendent entities) and are thus propositions which are to be characterized as being *a priori*, either analytic or synthetic, necessary, certain and universal then clearly one is adopting a philosophy of mathematics which is to be categorized as "realist".

2. The philosophico-mathematical problematic outlined above in terms of our three questions draws one immediately into the

very centre of the most disputed and controversial issues which are characteristic of contemporary philosophy. It should be obvious, from the preceding discussion, that this problematic raises immediately the central problems of contemporary theory of knowledge, philosophy of mind and metaphysics. But one should especially note that this problematic raises especially the most central and crucial themes of contemporary philosophical logic: issues such as those of the validity of the distinctions between the analytic and the synthetic and the *a priori* and the *a posteriori*, the problems of the clarification of the foundational semantic concepts of meaning, truth, necessity, reference, the issue of the preferability of the intensional versus the extensional approach. Thus one could argue that the best way of introducing beginners to the central themes of contemporary philosophy and of showing them the relevance of what might otherwise appear to be if not pointless at least extremely abstruse discussions might well be by means of allowing these issues to first emerge within the context of a consideration of the problematic which is handled by the philosophy of mathematics. What I am suggesting therefore is the possibility of employing the philosophy of mathematics as a way of introducing students to the wider issues considered by general philosophy rather than, as is usually done, relegating the philosophy of mathematics to the status of an elective within the philosophical curriculum and, at that, of an elective which more often than not is deemed to be of interest only to the specialist.

3. As I have already intimated in the first of these remarks, the diverse possible combinations of the responses which might be given to the three questions as to the objects, the manner and the nature of mathematical knowledge constitute various possible *philosophies* of mathematics. To mention but some of the principal ones, "realism" (or "Platonism"), "logicism", "intuitionism" (or "constructivism"), "foundationalist (Hilbertian) formalism", "non-foundationalist formalism", "conventionalism" and "quasi-empiricism". I shall not enter into the details of these various philosophies of mathematics at this point. I shall consider the more important ones in the second part of this paper where they will be presented from a historical point of view.

4. The philosophical problematic which is the concern of the philosophy of mathematics and which has been formulated in terms

of our three questions as to the objects, the manner and the nature of mathematical knowledge, as well as the diverse answers and combinations of answers (or distinct philosophies of materialistics) which have been proposed in response to them, must always be contextualized within the reality of the mathematical experience both past and present. Indeed it is the various undeniable elements which constitute this experience which ultimately enable one to adjudicate between competing philosophies of mathematics. It is to the extent that a particular philosophy of mathematics does, or fails to do, justice to these elements that is to be accepted or rejected as being an adequate philosophy of mathematics. I shall simply enumerate the most important of these elements: (1) The fact that *doing* mathematics consists largely of the rule-governed, formal, manipulation of symbol. (2) The fact that this seemingly mechanical manipulation of symbols is grounded in a rich dimension of subjective construction and daring intuition which is characteristic of the matematical consciousness. (3) The fact that, nonetheless, mathematical consciousness is not concerned with some psychologistically interpreted subjective dimension but with objective truths and structures. (4) The fact that mathematics holds of the real world and therefore must have some intimate connection with reality.

5. A final remark. In the following, historical, part of this paper I shall not be making any reference to the Thomistic philosophy of mathematics. This is not because I do not consider this philosophy of mathematics as worthy of consideration. But, as a matter of fact, the Thomistic philosophy of mathematics has been generally ignored, quite possibly because unknown, by the philosophical circles which have occupied themselves with the problematic dealt with by the philosophy of mathematics during the last one hundred years. Furthermore, even neo-scholastic circles have tended to neglect the Thomistic philosophy of mathematics. Certainly they have failed to attempt to reelaborate and express the Thomistic position in the terminology, the conceptual categories and in the light of the problems which have characterized contemporary discussions in the philosophy of mathematics. I believe that if these were to be done the Thomistic position would be revealed capable of making an important contribution and even as being the only philosophy of mathematics which can do justice to all

the elements constitutive of the mathematical experience enumerated in the preceding remark. Of course I cannot do justice to this contention in the present introductory context and, as it is worthy of separate treatment, I shall turn to it elsewhere.

(2) HISTORICAL INTRODUCTION TO THE PHILOSOPHY OF MATHEMATICS

The philosophy of mathematics of the last one hundred years or so has, at least until quite recently, been especially concerned with the problem of the *foundations* of mathematics. That is, it is in terms of this problem that the general philosophical problematic handled by the philosophy of mathematics (and which was presented in the preceding section in terms of the three questions as to the objects, the manner and the nature of mathematical knowledge) has tended to have been formulated, encapsulated. Just why has this been the case? Because the previously prevailing idea that mathematics is ultimately grounded upon *geometry,* upon our geometrical intuition of space and its properties, began to be rejected at the beginning of the second half of the nineteenth century thereby raising the problem of foundation.

1. The term "geometry" in this context means precisely *Euclidean* geometry. Now, there is little doubt as to the ultimately empirical origins of geometry and arithmetic in ancient Egypt and Babylonia. It seems certain that the origin of our primitive mathematical concepts was both experiential and inductive in character. This is indicated by the very term "'geometry", the *measurement of the earth.* Among the Greeks however, Thales of Miletus is credited with bringing geometry from Egypt to Greece around 600 B.C., mathematics commenced to be pursued for its own sake, for its purely theoretical or speculative interest. Both geometry and arithmetic began to be treated as purely abstract and *a priori* sciences dealing with abstract and ideal concepts in a manner completely independent of either their distant empirical origin or their possible practical applications. The development of Greek mathematics attained a certain climax with the composition of the *Elements* of Euclid (*floruit ca.* 300 B.C.) of whom little is known other than his membership of the Platonic Aca-

demy. This work is divided into thirteen books and treats of elementary geometry, arithmetic and in a certain sense even of algebra. It is thus a vast treatment of the entirety of Greek *elementary* mathematics. This work, because of its rigorous exemplification of the application of the axiomatic-deductive method, was destined to become for the next two thousand years not merely the basic textbook of geometry but also, and far more importantly, the exceptional model of how truly scientific thinking (demonstrative reasoning) ought to be structured. Nonetheless, it is not really an original and innovative work. Rather, it summarizes and systematizes the three preceding centuries of development of Greek mathematics. It also served as the point of departure for the subsequent development of Greek mathematics. The immediate successors of Euclid were Archimedes and Apollonius who no longer needed to deal with elementary mathematics but could devote themselves to questions of higher mathematics — for example, questions connected with the modern infinitesimal calculus (Archimedes) and the theory of conic sections, ellipses, parabolas and hyperbolas (Apollonius). After this time Greek mathematics entered into a decline, despite the brilliant exceptions of Ptolemy and Pappus, from which Western mathematics only began to recover during the sixteenth century.

From the philosophical point of view the most important of the thirteen books of the *Elements* is Book I. In this book Euclid introduces the greater part of the principles upon which the entire work is based. That is, the initial premises from which are derived, by way of deductive demonstrations, the more than five hundred theorems presented in the work. These principles are of three kinds: (1) twenty-three definitions and terms, (2) five postulates, (3) five axioms or common notions. Now, the foundational terms which are defined at the very beginning of Book I are all geometrical terms. Euclid only defines twenty-two arithmetical terms concerning the whole numbers at the beginning of Book VII and four arithmetical terms concerning the distinction between rational and irrational numbers at the beginning of Book X. Euclidean mathematics has therefore its supposed foundations in geometry whose own grounding principles (the definitions, postulates and the axioms common to all the sciences) are definitions and laws which presumably express linguistically the properties and relations which affect ideal geometrical entities (points, lines, triangles, etc.) which have an objective and transcendent ontological status and which the human mind discovers by means of a purely intellectual, entirely supersensible, intuition. Philosophically, therefore, the *Elements* might well be looked upon as the supreme mathematical manifestation of the philosophy of Plato: of its central realist doctrine of universals (the theory of separate forms or ideas).

It must also be stressed that it is precisely in the philosophical context of the Platonic Academy that Greek mathematics, as represented by the *Elements* and thereafter, became in the first place geometry and came to be considered as ultimately grounded in geometrical intuition. It is especially important to notice that this turn of events within the Platonic Academy represented a definite innovation with respect to pre-Euclidean Greek mathematics which had been predominantly arithmetical in character and had seen in arithmetic, rather than in geometry, its grounding discipline. This can be seen clearly from the emphasis placed upon arithmetic rather than geometry within the Pythagorean tradition. Now why did this happen? Just why did the Platonic Academy decide to ground mathematics upon geometry and thereby came to formulate a doctrine with respect to the foundations of mathematics which would remain largely accepted by the Western mathematical tradition well into the nineteenth century? It has been argued that this was the result of the Platonic desire to overcome the cosmological consequences of the discovery of the *irrational* numbers within the Pythagorean tradition. What are the irrationals and how did they come to be discovered? To understand this we need do no more than to consider the famous Theorem of Pythagoras: "the square of the hypotenuse of a right angle triangle equals the sum of the squares of the two other sides." But if the two sides of a right angled triangle, other than the hypotenuse, have, for example, the length "1" what will then be the length of the hypotenuse according to the theorem? It will clearly be the square root of 2. But just which number is the square root of two? It is a number which is not finitely determinable and which is called an "irrational" number because it is a number which cannot be represented as a relation between two whole numbers, as a fraction. In other words, it is impossible, and this as a matter of principle, to find a rational number of which the square will be exactly the number two. Thus the square root of 2 is 1.4142 . . . Such relationships which are expressed by irrational numbers are known as "incommensurables" and other well known examples are: (1) the relationship between the diagonal and a side of any square, (2) the relationship between the circumference and the diameter of any circle (pi). The consequence of this discovery of the irrationals was the ruin of the Pythagorean and Democritean, atomistic, doctrines which had sought to interpret the physical world in the purely arithmetical terms of the whole numbers. Because of this the Platonic Academy turned away from arithmetic to geometry insofar as in geometry the irrationals do not represent any kind of problem at all. Geometry itself came to be considered as being basically a theory of incommensurable proportions (irrational numbers) as opposed to arithmetic, the theory of rational numbers (wholes and fractions). Accordingly the Platonic tradition decided to treat all numbers geometrically and geometry came to be considered as the foundation of the entirety of mathematics. The *Elements* of Euclid therefore, might well be looked upon as being not only a textbook of geometry but as an instrument for the Platonic theory presented in the *Timaeus* which interprets the cosmos in geometrical rather than arithmetical terms.

2. The idea that mathematics is ultimately grounded in (Euclidean) geometrical intuition was generally accepted by the Western philosophical tradition long after the demise of Platonic metaphysics. Perhaps the most significant example of this is the way in which this idea was not only left unscathed but indeed reinforced by the famous "Copernican Revolution" operated by Immanuel Kant (1724-1804) in the field of the theory of knowledge and which, by contrast, had such detrimental effects upon so many other philosophical ideas that the Western tradition had treasured since the time of classical antiquity. The Kantian "Copernican Revolution", the hypothesis that the mind is active rather than passive, that in knowledge the mind does not itself conform to objects but rather contributes of itself certain *a priori*, transcendental, structures to which objects must themselves conform if

they are to be known at all, was meant to resolve the problem of the very possibility of the three theoretical sciences of the Western tradition, mathematics, natural science and metaphysics. This problem was reformulated by Kant, by means of his famous classification of judgments in terms of the four categorizations as *a priori* and *a posteriori*, analytic and synthetic, as the question of the possibility of synthetic *a priori* judgments (judgments which are no mere tautologies and at the same time both universal and necessary — and therefore neither verifiable on the basis of a mere analysis of the meanings of terms nor on that of sense experience) in each of these disciplines. The answer given by Kant to the question of the possibility of synthetic *a priori* judgments in the science of mathematics is to be taken as constituting his philosophy of mathematics.

In the "Transcendental Aesthetic", the first part of the *Critique of Pure Reason* (1781), Kant insists that the grounding propositions of mathematics are neither analytic *a priori* (and thereby he distantiates himself from both Leibniz and Hume for whom they were ultimately reducible to the principle of non-contradiction) nor empirical generalizations or synthetic *a posteriori*. Rather, they are, as must be the case with the foundational propositions proper to any authentic science, synthetic *a priori*. But just how is this possible? To answer this question Kant proceeds to isolate the *a priori* structures, the subjective transcendental elements of the faculty of sensibility. What *a priori* structures render possible the sensible perception of material objects through which they come to be presented to consciousness? What are the formal elements, inherent in the faculty of sensibility itself, under which is subsumed the material element, the manifold of sensation, thereby making perception possible to begin with? There are two of these and they are the *a priori* forms of sensibility which are *space* and *time*. Now, it is possible to attain *pure intuitions* of these *a priori* forms of sensibility by considering them in abstraction from their particular contents and it is on the basis of these pure intuition that we derive the synthetic *a priori* judgments which are the foundational propositions of mathematics. The pure intuition of space and its properties renders possible geometry and the pure intuition of time in addition to that of space renders possible arithmetic. Thus the grounding proposition of mathematics are synthetic because they are authentically informative insofar as they are descriptive of the *a priori* forms of space and time (and of the various mathematical objects, concepts, which can be constructed within the limits set by our pure intuitions of these) and, insofar as they are not attained inductively, are authentically universal and necessary. Thus for Kant the intuition of Euclidean space, geometrical intuition, grounds the entirety of mathematics just as it did for Euclid. The one essential difference is that whilst for Euclid geometrical intuition was ultimately of an objective *a priori* (transcendent *eidei*) for Kant it is of a subjective *a priori*, the transcendential structures of the faculty of sensibility.

3. The idea that mathematics is grounded ultimately in the geometrical intuition of Euclidean space nonetheless began to be rejected towards the middle of the nineteenth century. This took place because of several distinct but converging factors: (1) A very gradual rather than sudden factor was the continuous development of mathematical analysis — the theory of the continuum, of the infinite and the infinitesimal, essentially the infinitesimal calculus, integral and differential, which treats of variations of continuous magnitudes. This had begun in the seventeenth century and had been carried out by such great mathematicians as Descartes (1596-1655), Pascal (1623-1659), Fermat (1601-1665), Leibniz (1646-1716), Newton (1642-1727), James Bernoulli (1654–1705), Euler (1707-1783) ... The result of this continuous development was that by the middle of the nineteenth century the supposed grounding connection between geometrical intuition and analysis increasingly appeared to be remarkably distant and tenuous.

(2) The discovery and gradual elaboration of non-Euclidean geometries. This resulted from the questioning of Euclid's *fifth postulate* and the consequent elaboration of geometries founded upon its rejection by C.F. Gauss (1777-1855), N.I. Lobachewsky (1793-1856), J. Bolyai (1802-1860) and G.F.B. Riemann (1822–1866). This had a two-fold effect: (a) The general rejection of the idea that Euclidean geometry could serve as the foundation for the whole of mathematics because the basic concepts of Euclidean

geometry could no longer be considered to be necessary, truly universal and certain. (b) Insofar as the elaboration of the new non-Euclidean geometries took the form of their development as axiomatic systems this stimulated a movement towards the axiomatization of arithmetic itself. A movement of which the greatest steps would be taken by Giuseppe Peano (1858-1932), whose axiomatization of the system of the natural numbers was presented in his *Arithmetices Principia* of 1889, and by David Hilbert (1862-1943), who presented an attempt to axiomatize the entirety of mathematics in his *Grundlagen der Geometrie* of 1899, and which would find its culmination in the *Principia Mathematica* (I, 1911, II, III, 1913) of Russell and Whitehead.

(3) The gradual mathematization of formal logic which began with the algebraization of formal logic carried out by George Boole (1815-1864), and first programmatically presented in his *The Mathematical Analysis of Logic, being an Essay towards a Calculus of Deductive Reasoning* of 1847, and complemented especially by the work of A. De Morgan (1806-1871) and W.S. Jevons (1835-1881). As far as the problematic considered by the philosophy of mathematics is concerned this trend had the important effect of focusing interest on the affinity between mathematics and formal logic. An affinity which, as long as mathematics had been considered as ultimately grounded in geometrical intuition, had generally been ignored with the result that the two disciplines of formal logic and mathematics had been kept traditionally entirely separate. This trend would find its climax most especially in the works of Frege and of Russell, in that philosophy of mathematics known as "logicism".

4. The result of the convergence of these three factors was that mathematicians, being forced to abandon the traditional idea that mathematics is grounded upon geometrical intuition, turned towards arithmetic (elementary theory of numbers) to provide a foundation for the entirety of mathematics. To achieve this they sought, in the first place, to elaborate a unified theory of numbers. They tried to show how the more sophisticated theories which concern the more problematic kinds of numbers (irrational, real, imaginary, complex, transcendental) can all be reduced to the elementary theory which deals with the most fundamental kind of num-

ber, that of natural number. They attempted to do this by constructing the more sophisticated kinds of numbers by some process of definition which would have as its point of departure the natural numbers which were considered unproblematic and to be known intuitively on the basis of the everyday experience of counting. They then attempted to show that the various operations which can be executed within the ambits of the more sophisticated numbers can be reduced to the operations which can be executed within the ambit of the natural numbers. Finally, they attempted to show how the laws which govern the more sophisticated kinds of numbers can be deduced from the laws which govern the natural numbers. This development is referred to as the process of the *arithmetization of analysis* because it attempted to show how all the various parts of mathematics which may be classified as "analysis" could be reduced to the most elementary part of arithmetic, the theory of the natural numbers, and thereby rendered independent of geometrical intuition. This process was achieved during the second half of the nineteenth century as the result of the work of the great mathematicians of the period: Karl Weierstrass (1815-1897), Richard Dedekind (1831-1916) and Georg Cantor (1845-1918).

Now the essential theoretical component of this process was the construction through definition of the *real* numbers (the system of the rationals and the irrationals) in terms of the *natural* numbers. Each of these three great mathematicians proposed a technically different method for achieving this, yet they all relied upon the use of two fundamental notions: (1) the notion of *set* or *class,* and (2) the notion of the *infinite.* What this involved was a new interpretation of the mathematical *continuum* the fundamental nature of which would come to be understood in terms of arbitrary sets or totalities which were to be considered as actually infinite. The result of this was not only the successful definition of the real numbers in terms of the natural numbers by the employment of the mediating concept of *infinite sets* of *rational* numbers. It would also result in the tendency to define mathematics as "the science of the infinite" and in the fully elaborated theory of infinite sets which was worked out by Cantor as the theory of the *transfinite* numbers — a theory which would hold a central and controversial position in the philosophico-mathematical debates of the twentieth century. The achievement of this process of the arithmetization of analysis, of a unified theory of numbers, removed many of the philosophical perplexities which had been previously associated with the various more sophisticated kinds of numbers. This permitted the philosophico-mathematical problematic to be conceived increasingly in foundational terms and to be focused upon the fundamental question of the origin and clarification of the basic kind of numbers, the natural numbers.

5. The process of the arithmetization of analysis was accompanied by another, distinct though contemporaneous, process, that of the *axiomatization of arithmetic*, of the elementary theory of the *natural* numbers. The first attempt to achieve this, to reduce the theory of the natural numbers to the smallest possible number of undefined primitive terms or concepts and undemonstrated propositions or laws was made by Giuseppe Peano (1858-1932) in his *Arithmetices Principia,* 1889. He was the first to systematize the fundamental laws of the elementary theory of numbers in axiomatic form. Peano sought to show that the entire theory, of the natural numbers could be derived from, besides the concepts and laws of formal logic, only three fundamental concepts (undefined terms) and five basic axioms (undemonstrated laws). The three fundamental concepts are: (1) zero; (2) number; (3) successor. These concepts are not defined and the five axioms associated with them do not indicate their meaning. We are certainly not told whether they do or do not refer to or denote something real, they are left uninterpreted. The five axioms are: (1) zero is a number; (2) the successor of any number is a number; (3) no two numbers have the same successor; (4) zero is not the successor of any number; (5) every property which pertains to zero and which also pertains to the successor of any number that has that property, pertains to every number. The fifth axiom is the famous principle of mathematical induction. Now, if the five axioms are to be grasped as true propositions, the readers of Peano's treatise have to supply themselves a certain understanding of the three undefined terms. This was certainly presupposed by Peano who took it for granted that all would understand by "zero" the first element in the series of the natural numbers, by "successor" the number which immediately follows another number in the

same series and by "number" a member of that series. He took for granted therefore a certain intuitive grasp of the natural numbers common both to himself and his readers: at the very beginning of the elaboration of the axiomatization of the elementary theory of numbers all already know the meanings of these undefined terms and recognize the truth of the five axioms in which they appear.

The point is that, according to Peano, it is possible to demonstrate every proposition of elementary number theory by means of only these five axioms. Of course the laws of logical inference and deduction are also taken for granted, but on this matter there is no need to enter detail. Peano's axiomatic therefore represents the term of the process of the reduction of the entirety of mathematics to elementary arithmetic. It involves the systematization of arithmetic in an axiomatic form so that by then taking it as the point of departure the entirety of mathematics can be derived from it by means of construction through definition, and deduction (that is mathematical *induction*). First, on the basis of the axioms it is possible to define the elementary arithmetical operations of addition and multiplication and the general laws (commutative, distributive, associative...) which govern them. Second, in terms of these operations it is possible to define the inverse operations of subtraction and division. The problem of the extent of the possible performance of these operations within the ambit of the natural numbers results in the gradual enlargement of the numerical field. There is therefore the introduction of the negative and the rational numbers (the whole numbers and the fractions) which is followed by the introduction of the real numbers (the irrationals as well as the rationals) and finally by the introduction of the imaginary, the complex and the transcendental numbers. It is then possible to define the various arithmetical and algebraical operations of these new numerical systems. The entirety of mathematics is therefore shown to rest on Peano's three terms and five axioms and to be capable of being deduced from these by nothing other than the principles of formal logic.

It must be noticed though that, at least philosophically, Peano's treatise was far less satisfactory than might have seemed at first sight. Peano's work simply fails to give any kind of definitive resolution of the problem of the ultimate foundations of mathematical knowledge. In fact the work does no more than to avoid raising this issue. As we noted above Peano simply presupposes that we know already, from the very beginning of the process of the reconstruction of mathematics on the basis of his axiomatized number theory, both the meaning of the three fundamental terms "zero", "successor" and "number" and that the five axioms are true. But as a matter of fact his three primitive concepts can be interpreted in very many different ways each of which is perfectly consistent with ("satisfies") the five axioms. Thus Peano's axioms are not characteristic in any kind of exclusive way of the series of the natural numbers at all. Rather, they characterize the much wider concept of *progression in general.* Every progression whatsoever satisfies the five axioms of Peano and, as there is an infinite variety of progressions, it is possible to interpret Peano's system in an infinite variety of ways.

Therefore, the problem of the ultimate foundations of mathematics, far from being resolved, remains. A first attempt to resolve this problematic took the form of trying to define Peano's three terms in terms of concepts, and deduce his five axioms from law, which would be even more profound, namely, the fundamental concepts and laws of formal logic. This is the basic inspiration of the "logicist" programme in the philosophy of mathematics inaugurated by Frege and destined to be continued especially by Russell. Other attempts to resolve this problematic took the form of Brouwer's "intuitionism" which would claim that the three terms and the five axioms are known intuitively and *a priori* and Hilbert's "formalism" which would claim that the five axioms are not to be taken as representing absolute truths but are arbitrary rules for the combination of undefined variables which are not to be interpreted at all. But to these matters we shall turn in later sections.

6. Gottlob Frege (1848-1925) found his principal concern in the basic problem of the philosophical foundation of the unified theory of numbers that had been elaborated by Weierstrass and Dedekind, had been extended to the problem of the infinite by Cantor and was in the process of being axiomatized by Peano.

Frege approached elementary number theory from a far more philosophical perspective than any of his contemporaries. Frege insisted upon the identification of the theory of sets or classes as a part of formal logic. Therefore, given the central role played by the concept of set or class in the process of the arithmetization of analysis, for Frege the ultimate foundation of mathematics is attained by means of the elaboration of a unified theory of numbers which is based upon a concept of natural number which has been completely, that is, logically, clarified. This implied the reduction of the entirety of mathematics (arithmetic, geometry, analysis ...) to formal logic and the recognition of that discipline as being *the* foundation of mathematics. Thereby Frege became the inaugurator of the *logicist* programme and the founder of the school of *logicism* with respect to the philosophical problematic of the foundations of mathematics. For the logicist perspective mathematics is essentially a branch of formal logic. Therefore, (1) the basic concept of arithmetic, that of natural number, and through it all other numerical concepts can be reduced, by means of definitions, to purely logical concepts, and (2) the axioms of elementary number theory (arithmetic), as for example those elaborated by Peano, can be derived, by means of purely logical demonstrations, from purely logical laws. The central element in this entire procedure being the employment of the concept of set or class, taken as a logical concept, so as to arrive at a logical definition of the concept of natural number.

One should stress the originality that characterizes the Fregean programme. It represents the first attempt to unify two directions of mathematical thought which up to that time had rarely been seen as connected: (1) the technical mathematical element, corresponding to the work of Weierstrass, Dedekind and Cantor and their great predecessors of the eighteenth and seventeenth centuries; (2) the philosophical direction as represented, for example, by the radically opposed theories proposed by Kant and J.S. Mill (1806-1873). As far as these two opposed philosophical perspectives were concerned Frege rejected them both as entirely unsatisfactory. Frege rejected both the Kantian theory that mathematical propositions are synthetic *a priori* and Mill's contention that mathematical propositions are merely synthetic *a posteriori*, that they are no more than empirical generalizations which are ultimately derived from descriptions of the matter of fact, and hence contingent, working patterns of the human mind or brain insofar as it is accessible to empirical psychological research. Frege was thus totally opposed to both Kant's transcendental subjectivism and Mill's psychologism and stressed the objective character of mathematical knowledge by attempting to defend the thesis that mathematical propositions are all analytic *a priori* (with the exception of geometry which he held to be synthetic *a priori*), they are all reducible to the basic laws of formal logic — ultimately they are reducible, as had been argued by Leibniz, to the principle of non-contradiction.

It is important to notice that the formal logic to which Frege sought to reduce the elementary theory of numbers, and thereby the entirety of mathematics founded upon it through the arithmetization of analysis, was not traditional, Aristotelian, formal logic. Frege considered traditional formal logic as deficient in that he considered it to be affected by all the imperfections and ambiguities which accrued to it insofar as it was formulated in ordinary language. Rather, Frege sought to reduce mathematics to formal logic as it had begun to be mathematized by being treated algebraically in the work of Boole, De Morgan and Jevons. But Frege considered the actual work and achievements of these pioneers of symbolic or mathematical logic entirely inadequate. Therefore the first requirement posed by the realization of Frege's logicist programme was the full elaboration of an adequate logical language which would be both logically and conceptually perfect or at least as superior as possible to both traditional formal logic and Boolean logic. The elaboration of this linguistic instrument took the form of the system of symbolic logic presented in Frege's first major work, the *Begriffsschrift* of 1879. This work represents the beginning of truly modern symbolic logic and to be appreciated has to be recognized as the most important work in the history of logic since the *Analytics* of Aristotle. The contributions of this work which are of paramount importance include: (1) The first modern axiomatic-deductive systematization of the logic predicates and of the logic of proportions. (2) The introduction of the notion of propositional function. That is, the analysis of propositions of analogy with mathematical expressions into function and argument rather than into the subject, copula and predicates of traditional formal logic. (3) The introduction of the theory of quantification: the logical concepts of universal quantifier and of existential quantifier to express generality and existence respectively. These quantifiers being understood as predicates of predicates, as properties of concepts and not of individual objects. (4) The introduction of the notion of truth-function and of the distinction between use and mention.

Frege's first attempt to carry out his logicist programme with respect to the foundations of mathematics is presented in *Die Grundlagen der Arithmetik,* 1884. In this work Frege considers precisely the fundamental problem of the philosophy of mathematics, that of the definition of the concept of natural number. But it is treated largely from a polemical perspective and is directed against competing philosophies of mathematics. Frege's logicist programme is thus presented rather than achieved in this work. The polemical discussion, which constitutes the great bulk of the work, focuses on the various alternative and contemporaneous philosophies of mathematics: conventionalism: intuitionism; the empiricism of Mill; Kantianism. Frege is especially concerned with the refutation of the "psychologism" of Mill and other contemporary authors. He insists therefore on two points: (1) meanings are not to be confused with either sensible or intelligible private mental images — objects of consciousness are not to be confused with contents of consciousness; (2) logical and mathematical propositions must not be reduced to descriptions of matters of fact such as the modes of operation which are characteristic of the contingent human psyche, logic and mathematics cannot find their foundation in the empirical science of descriptive psychology.

During 1891-92 Frege published four articles which presented important clarifications and refinements of the basic logico-philosophical concepts essential to the realization of the logicist programme. (1) "Ueber der Traegheitsgesetz" 1891, deals with the principle of inertia. Frege takes the opportunity of considering the logical nature of the notion of *concept.* He distinguishes rigorously between a representation (*Vorstetlung*), a subjective fact considered by empirical psychology, and the concept (*Begriff*), a purely objective logical entity which as such is wholly external to any process of historical evolution. (2) In "Funktion und Begriff" 1891, "Ueber Begriff und Gegenstand" 1892 and "Ueber Sinn und Bedeuttung" 1892, Frege presents essentially a theory of meaning. That is, an exact determination of the concept of meaning for any linguistic element (expression) whatsoever (subject, predicate, preposition). Frege presents this theory of meaning in the context of a consideration of the cognitive value of analytic *a priori* propositions, are they informative or merely explicative (mere tautologies)? Frege's ultimate concern is precisely that of showing that mathematical propositions, which he considers to be analytic *a priori*, need not necessarily be mere tautologies but can be genuinely informative. One should notice that the analytic-synthetic distinction is taken by Frege in a sense quite different from the usual Kantian one of the relationship of inclusion or exclusion of the contents of the concepts of predicates and subjects within the unity of the proposition. For Frege the distinction between analytic and synthetic proposition is not founded at all upon their contents but upon their manner of verification. If a proposition can be verified without any reference to its content but in virtue of its logical form alone it is an analytic proposition but if it cannot be verified without reference being made to its content it is then synthetic.

For Frege, although all mathematical propositions are analytic *a priori* (with the exception of geometrical propositions which are synthetic *a posteriori*) they are not mere tautologies but indeed authentically informative. He attempts to show this by drawing the famous distinction between sense (*Sinn*) and reference or denotation (*Bedeutung*). The point of departure for his discussion is the attempt to establish whether identity is a relationship which concerns objects or the terms which are the names of objects. Thus in the expression "a = b" does the identity which is affirmed concern the terms "a" and "b" or the objects *a* and *b* which are named (denoted,, referred to) by those terms? Frege attempts to resolve this problem precisely by affirming that every expression has both a sense (intention, connotation, meaning) and a reference (extension, denotation). The sense of a term represents its manner of presenting the object that it denotes and which is its reference. In such a case of a singular term (logical subject, logical proper name) its sense or intention is an individual concept whilst its extension or reference is an individual object. The consequence of this distinction for propositions which affirm identity is clear. The expression "a = a" affirms the identity with itself of the object a denoted by the term "a" and is thus a trivial tautology. On the other hand the expression "a = b" whilst it also affirms the identity with itself of the object denoted by both the term "a" and the term "b" it also affirms something more. This is because even though the two terms "a" and "b" have the same reference, they denote the same object a, they have different senses and hence the expression "a = b" is not a tautology but can be truly informative. The relationship of identity therefore in the first place concerns objects. But in addition to this (extensional) identity another (intensional) identity might or might not be involved and it is on the facts of the presence or absence of this identity of sense that one can determine the cognitive value of any proposition which expresses identity, whether it be tautological or informative respectively.

In 1893 Frege published the first volume of *Grundgesetze der Arithmetik* (the second volume would be published in 1903 but the work as a whole was destined to be left unfinished). In this work he presented an effective attempt to realize the logicist programme of the reduction of arithmetic to formal logic by the employment of the conceptual notation and the refined logico-philosophical concepts developed in his previous works. Frege commences with a long introduction in which he takes up again the polemic against psychologism. This is followed by the presentation of the conceptual notation which had been developed in *Begriffsschrift.* Then there is the presentation of the fundamental concepts of arithmetic, especially that of natural number. Finally Frege presents the symbolic part of the work.

The argument presented in the *Grundgesetze* to define the concept of natural number is not greatly different from that which had been presented in the *Grundlagen* of 1884. But the concepts refined and developed during the intervening period enable him to express in terms of sets or classes (considered objects) the argument which he had previously expressed in terms of concepts. In other words, he now reformulates his argument in terms of an extensional, rather than an intensional, logic. We cannot, of course, enter into the technicalities of Frege's argumentation. Let it simply be said that the fundamental, and most original, element in Frege's discussion consists of the recognition of the possibility of expressing equivalence (or equipollence) between numbers without having to employ the concept of number itself. That is, in recognizing that given two classes, each one of which contains determinate elements (members), it is possible to establish whether to those two different classes does or does not pertain the same *number* of elements (that is, whether they are equinumerous classes) without having to employ any procedure of numeration or counting which would obviously presuppose the knowledge of the natural numbers and thus lead to a circular definition of the grounding concept of natural number. Frege achieves this by the introduction of the concept of the biunivocal relation of one-to-one correspondence. The employment of this notion enables him to arrive at a logically satisfactory definition of the concept of natural number and the identification of the natural numbers as a sub-class of the cardinal numbers, the finite cardinals. On the basis of the preceding logical definitions Frege subsequently carries out the successive derivations of the *principles of natural number*. That is, he deduces or logically demonstrates all the propositions which had been assumed as axioms by Dedekind and Peano in their axiomatization of elementary arithmetic. Thus Frege, in the *Grundegesetze*, appeared to have perfectly achieved his logicist programme of reducing the entirety of mathematics, which had been first arithmetized and then axiomatized by his contemporaries, to formal logic reconstructed and perfected as a symbolic or mathematical logic.

7. Bertrand Russell (1872-1970) was another important advocate of the logicist programme with respect to the foundations of mathematics. Russell came to hold a position extremely close to that of Frege although, at least at first, he had developed it quite independently of Frege's work, with which he became acquainted only after being directed to it by Peano whom he met at a philosophical congress held in Paris in 1900. From the very beginning Russell's philosophical interests were intimately connected, and would always remain in constant interaction, with his pre-

occupations with the problematic concerning the foundations of mathematics. This problematic, for Russell, corresponded to the issue of the character and ultimate justification of the truth of mathematical propositions. Russell was totally opposed to the extreme empiricism of J.S. Mill for whom the truths of mathematics were all synthetic *a posteriori* truths. Russell was always convinced that mathematical propositions are universal and necessary, analytic *a priori* truths. The problem which he felt confronted him was that of showing this to be the case.

As a student at Cambridge University during the early 1890s Russell had come under the influence of the then dominant neo-Hegelian idealism advocated by Ward, Stout, MacTaggart and Bradley and adhered to this philosophy until about 1894. Combined with his interest in the foundations of mathematics this early idealist philosophical commitment resulted in Russell's first important philosophical work, the *Essay on the Foundation of Geometry* of 1897. It defended an essentially neo-Kantian position. Yet this work was almost immediately repudiated by Russell. This was because, around 1898, Russell broke definitively with idealism as the result of the influence exercised upon him by G.E. Moore (1873-1958). In reaction against his former idealism Russell adopted the contrary position of extreme, almost Platonic, realism and for a while was particularly sympathetic to the doctrines of Alexius von Meinong (1853-1920). In the field of the philosophy of mathematics a general philosophical realism implied the adoption of logicism.

Russell first presented his logicist philosophy of mathematics in the *Principles of Mathematics* of 1903. As delineated in this work Russell's conception of the logicist programme with respect to the foundations of mathematics was similar to that of Frege. It consisted of two basic steps: (1) the reduction of the entirety of mathematics to certain basic propositions, laws, concerning the natural numbers — this corresponded to the work already carried out by Weierstrass, Dedekind and Cantor and which has been axiomatized by Peano; (2) the reduction of these basic propositions concerning the natural numbers to the basic propositions of a system of formal logic — this corresponded to the work already carried out by Frege but, during the period of the composition of the *Principles,* Russell was not yet aware of this. Thus, the second of these two steps represented Russell's own contribution and it had appeared to him as involving the prior fulfilment of two requirements: (1) the discovery of a method of defining the natural numbers in purely logical terms; (2) the development of a system of symbolic logic which would be adequate to the task of deducing by means of its employment all the basic laws of arithmetic from the basic laws of a system of formal logic.

One should note that the fulfilment of the second of these two requirements corresponds to the task of the mathematization of formal logic, that is, of treating formal logic as an algebraic and a symbolic system. One must be careful, at this point, not to confuse this procedure with the logicist programme itself of reducing the entirety of mathematics to formal logic. They are obviously in no sense to be identified. Thus, one can well accept the notion of a mathematical or symbolic logic without having at all to adhere to the logicist philosophy of mathematics, to the philosophical thesis known as "logicism" that the foundations of mathematics are constituted by formal logic. The notion of a mathematical logic is of itself equally compatible with any other philosophy of mathematics whatsoever.

Russell's attempt to fulfill the first of these two requirements took the form of his definition of the notion of natural number itself. At this point Russell employed the same concept which had been employed by Cantor and Frege to explain the concept of equinumerosity of classes without employing the concept of number, he employed the concept of the biunivocal relation of one-to-one correspondence. Accordingly Russell commenced with the definition of zero. Zero is the class of all classes which have as many members as the class of objects which are not identical with themselves. Because these would be contradictory objects which as such could not possibly exist, the class of objects which are not identical with themselves is empty — it is the null class. Zero is thus defined as the class of empty classes. But as there is no difference whatsoever between one empty class and another empty class, in line with the Leibnizian principle of the identity of indiscernibles, there must be only *one* empty class, *the* empty class. The class of all empty classes, which is the number zero, has therefore only one member, itself. The number one is then defined by reference to the fact that the class of all empty classes has only *one* member. That is, the number one is defined as the class of all classes which are equinumerous with the class of all empty classes, as the class of all classes which have as many members as that class which is the number zero. The number two is then defined as the class of all classes which have as many members as the class whose members are the numbers one and zero, and so on to infinity.

Russell's attempt to fulfill the second of these two requirements took the form of the elaboration, carried out in collaboration with A.N. Whitehead, of the great system of mathematical logic which would be presented in *Principia Mathematica* I 1910, II and III 1913. Now, Russell and Whitehead had commenced working on *Principia Mathematica* as early as in 1901. It was whilst working on this first volume that in 1902 Russell discovered a logical contradiction or antinomy which he seemed to affect the notion of set or class itself and thus appeared to place in peril the entire logicist programme as it implied, at least at first sight, the self-defeating because contradictory nature of any attempt to define the concept of natural number by means of the employment of the notion of class.

8. At this point it might be useful to interrupt this account and cast a backyard and general glance upon what had been happening in the philosophy of mathematics during the second half of the nineteenth century. One could well say that there had seemed to be a convergence of originally and intrinsically quite diverse elements towards the elaboration of a new and rich philosophico-mathematical synthesis which appeared destined to become definitive. Let us summarily call them to mind: (1) The arithmetization of analysis, the elaboration of a unified theory of numbers as the basis for the entirety of mathematics. This elaboration had its essential element in the definition of the real numbers in terms of the natural numbers through the employment of the concept of infinite sets. This was achieved by the work of Weierstrass, Dedekind and Cantor. (2) The axiomatization of this unified theory of numbers. This was initiated by Peano and further stimulated by the axiomatization of both Euclidean and

non-Euclidean geometries effected by Hilbert. (3) The elaboration of mathematical or symbolic logic which, inaugurated by Boole would come to a climax in the work of Frege and Russell. (4) The emergence of the philosophical school of logicism, represented first and foremost by Frege and Russell, which attempted to provide a truly ultimate foundation for mathematics by means of the reduction of this newly elaborated, and axiomatically treated, unified theory of natures and thereby the entirety of mathematics to formal logic treated mathematico-symbolically.

Now, these four elements appeared destined to become unified within that definitive synthesis represented by the projected *Principia Mathematica* on which Russell and Whitehead commenced working in 1901 — a work which was meant to achieve for mathematics what Newton's *Principia Mathematica Scientiae Naturalis* 1687 had achieved for classical, mechanistic, physics. Thus, at the turn of the century everything seemed to be proceeding smoothly and there was a general feeling of optimism and even self-satisfaction. But it was precisely at this point, in 1902, that Russell discovered that the then current theory of sets or classes—the element which, as I have repeatedly stressed, was meant to be foundational to the entire structure of mathematics—was incoherent insofar as it was vitiated by certain internal contradictions. As a matter of fact the antinomy discovered by Russell was neither the first nor the only contradiction discovered to affect the notion of class but it was the one which captured the attention of mathematicians as being particularly momentous and the harbinger of disaster. This quite unexpected turn of events left two of the four elements mentioned above quite undisturbed: (1) the continuing development of mathematical or symbolic logic; (2) the tendency towards the axiomatization of arithmetic and thereby the entirety of mathematics. Indeed the tendency towards axiomatization would during the first half of the twentieth century be extended to set theory itself and result in the great achievements of E. Zermelo, A. Fraenkel, T. Skolem... But it appeared to destroy completely the two other elements: (1) the conviction that the entirety of mathematics could be founded upon arithmetic by means of the definition of the real numbers in terms of the natural numbers through the employment of the notion of infinite classes; (2) the logicist programme which by means of the logical definition of the concept of natural number in terms of the concept of class taken as a logical concept attempted to reduce the entirety of mathematics to formal logic as being its foundational discipline. The combination of the apparent collapse of these two elements constitutes the famous "crisis of foundations" which was generally held to affect mathematics during the opening years of the twentieth century.

Just what was the contradiction discovered by Russell in 1902? It is known as "the paradox of the class of all classes which are not members of themselves". It implied, at least at first sight, the contradictory nature of any attempt to define the concept of natural number by means of the notion of class. This, because to be able to proceed as had been done by both Frege and Russell, it seemed to be necessary in order to give a logical definition of the concept of natural number by means of the notion of class to be able to continue, and this as a matter of principle, to form classes of classes, classes or classes of classes and so on to infinity without any restriction whatsoever. That is, all possible classes must themselves be capable of classification, of becoming members in their turn of still further classes. The problem emerged as to whether this was really possible and the test-case was clearly whether a class could be a member of itself.

Now it is clear that most classes cannot be members of themselves. The class of philosophers, for example, is not, unlike Gassendi and Malebranche, itself a philosopher, and hence cannot be a member of the class of philosopher, cannot be a member of itself. But some classes do seem to be able to be members of themselves. Thus, for example, the class of all catalogues is itself a catalogue and hence seems able to be a member of the class of all catalogues, seems able to be a member of itself. It appears then that classes are of two kinds, that classes themselves fall into two classes: (1) the class of classes which are not members of themselves; (2) the class of classes which are members of themselves. But the first of these two classes, the class of classes which are not members of themselves, is itself a member of itself or is it not? Clearly if this class *is* a member of itself then, since it is precisely the class of classes which are not members of themselves, it *cannot* be a member of itself. But, on the other hand, if this class *is not* a member of itself then, once again since it is precisely the class of classes which are not members of themselves, it *must be* a member of itself. But this is clearly contradictory or, to put it more benignly, "paradoxical".

Now, one must notice that this paradox is not at all as trivial as it might appear to be at first sight. For a proper appreciation of the paradox's significance one needs to stress what must be denied or rejected as the result of its discovery. That is, it prohibits as problematic any attempt to move from a concept to its extension. It implies a prohibition with respect to taking for granted the principle that every property defines the class or set of elements which verify it, the totality of elements of which that property holds true — its extension. Indeed, it implies that the principle that every natural property determines a class of things which satisfy that property is contradictory.

As a result of the discovery of this paradox, which had been communicated to him immediately by Russell, Frege himself decided to abandon the logicist programme altogether and leave the *Grundgesetze* unfinished. He could not see how one could possibly save the logicist programme once its grounding concept of class had been shown to be contradictory and hence unemployable. In his later years Frege did eventually return to the problematic of the foundations of mathematics but did so from a neo-Kantian rather than a logicist perspective. But, in addition to this rather negative Fregean reaction, there were to be three great responses to the crisis in the foundations of mathematics occasioned by the discovery of Russell's paradox. These three responses constitute the three principal philosophies of mathematics formulated during the first half of the twentieth century: (1) continuing (nominalistic) logicism; (2) intuitionism; (3) formalism.

9. The first response, which might best be described as taking the form of a continuation of the logicist programme as recast in a nominalist perspective, was Russell's own. Basically Russell attempted to save the logicist programme of reducing the entirety of mathematics to formal logic, by means of the definition of the concept of natural number in terms of the concept of set taken as a logical concept, by the introduction of a number of steps which

he believed would enable him not so much to overcome as to avoid the very emergence of the famous paradox that he had himself discovered. In the first place, around 1904, Russell's though underwent a third major change of direction (the first had been his adoption of idealism around 1890, the second his rejection of idealism and adoption of extreme realism around 1898). That is, he rejected his former extreme realism in favour of a far more moderate realism which would gradually but steadily transform itself into a thorough nominalism by the time of the publication of the 2nd edition of *Principia Mathematica* in 1928. This movement towards nominalism was not only occasioned by factors intrinsic to Russell's attempt to continue the logicist programme such as those that I shall refer to presently. It was also the result of the increasing influence exercised upon Russell's thought by Ludwig Wittgenstein whose fundamental theses, presented in the *Tractatus Logico Philosophicus*, that the propositions of logic and mathematics are analytic *a priori* and hence mere tautologies and the logical constants are not denotative, would be accepted by Russell himself by the beginning of the 1920s.

Russell's rejection of his former extreme realism around 1904 was due, according to Russell himself, to the development of a new sense of reality which rendered him increasingly sceptical with respect to extreme realism and its proliferation of queer entities such as non-existent and contradictory objects. This is witnessed by Russell's increasingly critical attitude with respect to the thought of Meinong whom he had previously greatly admired and on whom he published a number of hostile articles in *Mind* in 1904 and 1905. But far more important was perhaps his realization that the paradox concerning the concept of class which he had discovered, and which had seemed so dangerous for the logicist programme, had arisen out of his and Frege's extreme realist interpretation of classes as objects. Russell's various attempts to resolve the paradox accordingly took the form of the development of certain analytic techniques which would enable him to reject his previously extreme realist interpretation of the notion of class and to increasingly distantiate himself from extreme realism as a general philosophical position.

The techniques of analysis in question were principally two: (1) the theory of types; (2) the theory of definite descriptions.

In the present context I cannot enter into the details of these two theories but two general remarks might well be worth making. (1) The development of both these techniques was rendered possible by the application of a number of fundamental logical notions that Russell had developed, and in the development of which he had been largely anticipated by Frege, as the result of the attempt to produce a system of symbolic logic which would be adequate to the realization of the logical programme. For example, the notions of "truth-function", "propositional function", "universal quantifier", "existential quantifier" and so on. (2) The development of these two analytical techniques carried philosophical implications which would go well beyond the narrow confines of the philosophy of mathematics and result in the elaboration of a new conception of analytic philosophy itself. Russell's conception of philosophy would become not that of a mere clarificatory analysis of ordinary language in line with the dictates of common sense, as had been Moore's, but rather that of a reconstructive analysis concerned with the elaboration of an ideal language in the form of a system of mathematical logic. Thereby Russell would become the founder of a central orientation within analytic philosophy which would number among its adherents such philosophers as the Wittgenstein of the *Tractatus Logico-Philosophicus,* the great majority of the Logical Positivists and especially Rudolf Carnap and W.V.O. Quine.

Russell's proposed solution to the paradox of the class of all classes which are not members of themselves took at first the form of the so-called "simple theory of types." Russell was convinced that the source of the very emergence of the paradox was the unjustified conviction that all classes could be treated as arbitrarily classifiable objects. But he argued that this was not the case at all because classes and individuals are of radically different logical types and what can be false or true of one logical type need not even be meaningful when asserted or denied of another logical type. Thus, according to Russell, an assertion such as "the class of philosophers is a philosopher" ought not to be regarded as false so much as meaningless. Therefore what can be significantly affirmed of individuals cannot be significantly affirmed of classes and vice versa, and so on through the hierarchy of logical types. Thus if one were to respect rigorously the difference between the different levels of the hierarchy of logical types, the paradox of the class of all classes which are not members of themselves (as well as the various other paradoxes affecting the notion of class) would not even arise because it could not be significantly formulated insofar as the self-reference which gives rise to it would have been inhibited.

In the light of this prohibition with respect to the endless formation of classes of classes Russell then proposed a new manner of defining the concept of natural number in terms of the concept of class. He kept the definition of zero as the class whose only member is the null-class, but he now treated the number one as the class of all classes which are equinumerous with the class whose members are: (1) the members of the null-class; (2) any object not a member of that class. The number two was then defined as the class of all classes whose members are: (1) the members of the class used to define the number one; (2) any object not a member of that class. In this way Russell proposed to define the numbers one after another and each number would be a class of individuals rather than of classes. But then the series of the natural numbers could be continued *ad infinitum* only if there were an actually infinite number of objects in the universe, otherwise the series would end sooner or later. Russell perceived this implication and accordingly added the famous *axiom of infinity,* the hypothesis that the number of objects actually to be found in the universe is not finite but infinite. But then this axiom is hardly a truth of formal logic so that its adoption represented a kind of admission of the impossibility of realizing the logicist programme in its original conception as wholly free from extra-logical, empirical, elements.

In fact, despite the undeniable greatness of *Principia Mathematica* as a work of mathematical logic, the entire history of Russell's continuing logicisms cannot be looked upon as anything other than a succession of more or less *ad hoc* measures of repair all of which eventually proved unsatisfactory and at times even responsible for the introduction of even greater difficulties than those that they were intended to resolve. Thus, as we have seen the new attempt at defining the concept of natural number necessitated the introduction of an existential postulate, the *axiom* *of infinity.* Similarly the attempt to deduce the laws of arithmetic from purely logical laws necessitated the introduction of a further existential postulate, the *axiom of choice.* Perhaps the central problem was that of the definition of the *real* numbers and this problem interacted with the continuing need to resolve the various paradoxes affecting the concept of class, with what came to be called the problem of "impredicable properties." To resolve the problem of the definition of the real numbers Russell tried to *construct* them but this resulted in the emergence of the fundamental problem of *impredicative definitions.* That is the problem of the permissibility of definitions in terms of the totality of which that which is to be defined is a member — that is, the definition of the real numbers in terms of the set of all real numbers. Russell rejected impredicative definitions and formulated an injunction against their employment (the "vicious circle principle"). To be able to observe this injunction he was forced to modify the simple theory of types, which had been elaborated to resolve the paradoxes affecting the concept of class as we have seen, and formulate the *ramified theory of types.* But this ramified theory of types itself resulted in further difficulties in the theory of real numbers which induced Russell to introduce yet another axiom, the *axiom of reducibility* which itself proved to be so problematic that Russell abandoned it by the time of the publication of the 2nd edition of *Principia Mathematica* in 1928.

What was the concluding nominalist logicist conception of mathematics? Mathematics came to be seen as having quite literally no subject matter, as not being about anything at all. Mathematics consists of a body of truths which are analytic *a priori* truths because they can be derived by deduction, and the concepts which are their content by definition, from the laws of formal logic. Mathematical propositions are true in virtue of their logical form alone.

Except for the insistence on the absolute truth of mathematics and its grounding in formal logic the nominalist logicist conception of mathematics became increasingly that of a *formal* axiomatic. As such the logicist programme's foundationalist aspirations of Hilbertian formalism. But to this we shall turn below.

10. The second response to the crisis in the foundations of mathematics occasioned by Russell's discovery of the paradox of the class of all classes which are not members of themselves took the form of that philosophy of mathematics known as "intuitionism" (and also as "constructivism" and "conceptualism") and which was first proposed by the Dutch mathematician L.E.J. Brouwer (1881-1967) in his doctoral thesis of 1908. The intuitionism of Brouwer is admittedly a position which is ultimately Kantian in inspiration. According to it the natural *numbers* are known in virtue of an original and natural intuition and this intuition is the sole acceptable point of departure for the entirety of mathematics. Furthermore, commencing from this original and natural intuition of the natural numbers, the whole of mathematics must then be arrived at by a process which is to be thoroughly *rigorous*. That is, a process of construction through definition which, as a matter of principle, must consist of a *finite* number of steps.

For the intuitionist mathematics is clearly about something, about a certain kind of objects. But these mathematical objects are most certainly not to be affirmed as being any kind transcendent entities in a realist sense. In fact, in line with the Kantian affirmation of the matter of principle unknowability of the *Ding-an-sich,* the question of the extra-mental reality of mathematical objects cannot legitimately be raised. The objects of mathematics are thus mathematical concepts or categories which are the products of human thought and exist only in human thought. Because mathematics is about such mental concepts formed by a finitistic process of construction mathematical propositions are for the intuitionist synthetic *a priori* as they had been for Kant. But one must note that for intuitionism not all mathematical propositions need be either true or false in themselves. The intuitionist perspective leads to the conclusions that there are three types of mathematical propositions: (1) those which are true; (2) those which are false; (3) those which are undecidable. Now it must be stressed that intuitionism argues that the last group of propositions, those which are undecidable, are not so just on account of some subjective and possibly only temporary limitation currently affecting our mathematical knowledge. Rather, they are undecidable in themselves. For example, the famous conjectures of Fermat an Goldbach are usually indicated by intuitionists as being undecidable and hence neither true nor false in themselves.

The greatest problem which faced the intuitionist philosophy of mathematics was that it was forced to reject a great part of classical mathematics and especially the theory of transfinite numbers which had been elaborated by Cantor. This followed from its insistence upon the rigour, the finitist character, which must pertain to the constructivist process.. More particularly, because the intuitionist could not employ any kind of mathematical reasoning, demonstration or proof, which made use of reductions *ad absurdum*. This, because such reductions would have to presuppose the universal validity of the principle of the excluded middle. But it is precisely the validity of the principle of the excluded middle which is rejected by the intuitionist in virtue of his adherence to the thesis that there are mathematical propositions which are neither true or false, but undecidable, in themselves. Furthermore it should be noticed that it is precisely the rejection of the universal validity of the principle of the excluded middle which constitutes the meaning of intuitionism's *anti-realism*. Because of its rejection of much classical mathematics and the Cantorian transfinites intuitionism has not had a great number of adherents being limited, generally speaking, to the immediate disciples of Brouwer. But it must be mentioned that in recent years the intuitionist philosophy of mathematics and intuitionist logic in particular, as well as the extension of its central anti-realism to general philosophy, has received a particularly keen defence in the works of Michael Dummett especially his *Elements of Intuitionism* 1978.

11. The third response to the crisis in the foundations of mathematics occasioned by the discovery of Russell's paradox must also be seen as a reaction against the "infinitistic iconoclasm" of Brouwer's intuitionism. This response, known as "formalism" was elaborated by David Hilbert (1862-1943). Hilbert's formalism is a philosophy of mathematics which, accepting the impossibility of reducing the entirety of mathematics to formal logic because of the paradoxes affecting the concept of class, attempts to

find mathematics upon itself by means of the elaboration of a metamathematics. This metamathematics is conceived of as a purely formal system, as an empty axiomatics (that is, an axiomatics whose terms are left uninterpreted) and of which Hilbert sought to demonstrate the consistency (that is, the absence of internal contradiction) by means of finitist and constructivist methods which would be acceptable even from the point of view of Brouwer's intuitionism. The entirety of mathematics is then founded upon this metamathematics insofar as the originally non-interpreted terms of this purely formal axiomatic system are then interpreted by being given the meanings associated with the various mathematical concepts and structures.

Now the foundationalist aspirations of Hilbertian formalism, as well as the foundationalist aspirations of the logicist programme, were completely destroyed by the famous proofs presented by Kurt Goedel in 1931 which demonstrated the impossibility, as a matter of principles, of elaborating a formal axiomatic system which could be at the same time both consistent and complete and, in 1933, which demonstrated the impossibility of a finitary consistency proof of number theory. (See K. Goedel, "On Formally Undecidable Propositions of Principia Mathematica and Related Systems," 1931, "On Intuitionistic Arithmetic and Number Theory," 1933. Both reprinted in Davis, M. (ed.), *The Undecidable*, New York: Raven Press, 1965).

Nevertheless, despite the destruction of the foundationalist' aspirations of Hilbertian formalism, formalism did become from about the beginning of the 1930s and remained until very recently the dominant philosophico-mathematical perspective. Of course this now dominant formalism no longer adhered to the foundationalist aspirations of the original Hilbertian programme. Perhaps the character of this non-foundationalist formalism is best represented by the famous series of mathematical textbooks published in France under the collective pseudonym of Nicolas Bourbaki. What are its essential elements?

Mathematics is literally not about anything, there are no mathematical objects either in the sense of realist transcendent entities or in the sense of conceptual entities, categories. Mathematics is no more than a rule-governed, formal manipulation of symbols, or rather of mere signs or characters which, in fact, do not symbolize anything. Mathematics is thus of itself an empty or non-interpreted formal axiomatics. This empty formal axiomatics consists of a series of rules for operations which do not treat of any objects whatsoever. Mathematics is therefore to be defined as the science of the structures of formal deductions of theorems from axioms. In this context mathematical *existence* means quite simply the possibility of consistent definition, the absence of contradiction, within the formal axiomatic. As there are in fact no mathematical objects to be known in either a realist or a conceptualist sense the problem of mathematical *intuition* or *access*, the problem of the manner of our knowledge of mathematical objects, does not even arise for formalism. The nature of mathematical knowledge is such that all mathematical propositions are clearly analytic *a priori*, they are ultimately tautologies the meanings of whose terms are quite arbitrary and fixed by convention. Furthermore, these mathematical propositions are only true by convention, they are certainly not to be taken as being either true or false in any kind of absolute sense. The "truth" of mathematical theorems is no more than their consistency with other theorems and with the axioms from which they have been deduced. It should be particularly noticed and stressed that in the formalist perspective the supposed axiomatic-deductive method which for the Western tradition has generally being considered to be characteristic of the *a priori* science of mathematics has been thoroughly assimilated to the hypothetico-deductive method traditionally characteristic of the natural, empirical, sciences. For formalism the grounding axioms of mathematics are neither evident nor certain, let alone true in any absolute sense, they are no more than mere hypotheses.

12. The three great schools of logicism, intuitionism and formalism do not exhaust the entire spectrum of philosophical opinions constitutive of the philosophy of mathematics of the twentieth century. Thus realism, usually in a demythologized rather than a literally Platonic form, has continued to be represented and at times and in certain mathematical contexts has even come to the fore. To give but two examples, the set-theoretic Platonism associated with the axiomatization of set theory carried out during the first few decades of the present century by such authors as Zer-

melo and Fraenkel and the avowed Platonism of such a renowned destroyer of foundationalist programmes as Kurt Godel. But a general remark that can be made about the philosophy of mathematics of the twentieth century is that until very recently it has found its principal preoccupation in foundationalist questions.

In this regard the last two decades have witnessed the development of a tendency towards a radical change of orientation in the philosophy of mathematics. A new direction in the philosophy of mathematics has emerged which is stressing the lack of success of all previous foundationalist programme (realism, logicism, intuitionism, formalism) and which is even asserting that the philosophy of mathematics ought not to find its principal concern, let alone its exclusive concern, in foundationalism. This trend in the philosophy of mathematics, following the usage of possibly its most important representative, Imre Lakatos (1922-1974), is increasingly being referred to as "quasi-empiricism". This new trend has developed out of two factors: (1) an ever increasing interest in the history of mathematics which has shifted attention away from the abstract problems of the order of mathematical justification to the order of actual mathematical discovery and mathematical practice; (2) the ever increasing assimilation of mathematics to natural, empirical, science, as the result of the formalist's identification of the axiomatic-deductive method of mathematics with the hypothetico-deductive method. The "quasi-empiricist" approach in the philosophy of mathematics has accordingly stressed the inductive and empirical, *a posteriori*, character of mathematical knowledge. It has also stressed, more often than not from a Popperian perspective, the fallibility and corrigibility which have been associated with all the advances which have been made in mathematical knowledge. Finally, it has stressed the importance of looking at actual mathematical practice and of understanding the mathematical experience in its light rather than of theorizing and legislating in an *a priori* manner about what mathematicians ought to be doing and how they ought to go about it.

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BIBLIOGRAPHICAL NOTE

This note is meant not only to acknowledge my sources but also to provide a first bibliographical orientation for beginners who might desire to continue their reading in the philosophy of mathematics. Accordingly, I shall mention only introductory and general literature which is likely to be accessible to such readers rather than specialist literature.

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